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PROPOSING 'ONAHDO' SINUSOIDAL SERIES FOR THE EXPANSION OF THE NTH ORDER SINE FUNCTIONS WITH SOME APPLICATIONS FROM LITERATURE

OGWUMU ONAH DAVID

Mathematics and Statistics Department, Federal University Wukari, Nigeria.

ABSTRACT

The study proposed an algorithm for expanding the nth order Sine Functions via the knowledge of Demoivre's and Binomial Theorems. Similarly, the polar form of the complex Number, $Z = u + iv$ and its associated conjugate $\bar{Z} = u - iv$ (where, $i = \sqrt{-1}$) was employed operating in a unit circle. Thereafter, the

INTRODUCTION

In real-life computation and practice, the calculus of integration and sometimes differentiation of an nth power trigonometric function (like $Sin^m\theta$) is actually a difficult task [1, 2, 5]. This is especially when the power of the trigonometric function is very large (i.e., when n = 8, 9,….40,... etc). In such a situation as this, the researcher would definitely resort to using rigorous substitution and reduction formula in order to achieve his/her objectives. Hence, the need for a method/algorithm/sequence of approaches for transforming the nth power Sine function into a linear version of the nth degree function cannot be overemphasized [3].

For instance, in the work of [4], [5] [6] and [7], to integrate the Third, fourth, fifth or other higher order of sine and cosine functions, the task could involve

series derived in the study was tested on few examples in literature. And the test confirmed that the algorithm proposed is suitable in handling the family of problems considered. This is because, the algorithm required fewer steps as compared to some approaches like reduction formula and substitution formula for integration any nth or power Sine Functions. It was further observed that the algorithm could be suitably used for such array of problems in real life.

Keywords: Binomial Theorem, Demoivre's Theorem, complex Analysis, complex conjugate, n^{th} order, Sine function.

the use of several reduction formulas, expansion, assumption and enormous substitution tasks.

Thus, in this study, we proposed a series of algorithm as reported in [8] for breaking down of an n^{th} order sine or cosine function into a set of sub-functions of the form:

$$\left.\begin{aligned} Sin^n\theta &= aSin\theta + bSin\theta + \cdots \\ Cos^n\theta &= aCos\theta + bCos\theta + \cdots \end{aligned}\right\} \qquad (1)$$

But, in order to derive the algorithm proposed in this study, we assert that in basic trigonometry/algebraic expansions for function like $Cos^2\theta, Cos^3\theta$ and $Sin^2\theta, Sin^3\theta$ could easily be derived such that we could have;

$$Cos^2\theta = 2^{-1}(1 + Cos2\theta); \quad Cos^3\theta = 2^{-2}(Cos3\theta + 3Cos\theta);$$

$$and;$$

$$Sin^2\theta = 2^{-1}(1 - Cos2\theta); \quad Sin^3\theta = 2^{-2}(-Sin3\theta + 3Sin\theta)$$

But in this study, our attention is centred on deriving an algorithm for the 4^{th}, 5^{th} ….. up to the n^{th} order of any Sine and Cosine function.

However to achieve this, we recall from the definition of complex number that as used by [9] that; $Z = x + iy$

Derivation of the Sinusoidal 'ONAHDO' Series

In this section, the study, addressed the following subsections:

- Derivation of the Preliminary Equations
- Derivation of the Even order Sin Functions Algorithm
- Derivation of the odd order Sin Functions Algorithm

Derivation of the Needed Preliminary Equations

Accordingly, from the Demoivre's Theorem, the polar form of any complex variable and function $Z = x + iy$ as used by [10] for any r as radius of a circle and n being an integer or real number and $i = \sqrt{-1}$ then we have;

$$\left.\begin{array}{l} z^{n} = r^{n}[Cos(n\theta) + iSin(n\theta)] \\ z^{-n} = r^{n}[Cos(n\theta) - iSin(n\theta)] \\ \\ \left.\begin{array}{l} \therefore z^{n} + z^{-n} = 2Cos(n\theta) \\ \therefore z^{n} - z^{-n} = 2iSin(n\theta) \end{array}\right\} where\ r = 1 \end{array}\right\} \quad (2)$$

Therefore, since the focus of this study is on the expansion of sine and cosines of higher degrees (from 4th order and above), then from equation (2) above, When n=1, and r =1 (in a unit circle), we have that;

$$z = Cos\,\theta + iSin\,\theta \quad (3)$$

$$z^{-1} = Cos\,\theta - iSin\,\theta \quad (4)$$

Similarly, subtracting equation (4) from equation (3), gives;

$$\left.\begin{array}{l} 2i\sin\theta = z - z^{-1} \\ Sin\theta = (2i)^{-1}\left(z - z^{-1}\right) \end{array}\right\} \quad (5)$$

$$\therefore Sin^{n}\theta = (2i)^{-n}\left(z - z^{-1}\right)^{n} \quad (6)$$

As mentioned earlier, the results for when n = 2 and 3, the result of the expansion in equation (6) is easily gotten. But now, we can use equation

(5) and (6) to expand sine functions of higher degrees when the powers of $n \geq 4$ as given in the subsections below.

Derivation of the Even order Sine Functions Algorithm

From the above equation (6), when n=4, it implies that;

$$\sin^4 \theta = (2i)^{-4}\left(z - z^{-1}\right)^4$$

Hence, Using the binomial expansion to expand the above equation yields,

$$\sin^4 \theta = 2^{-4}\left[z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}\right]$$

$$\sin^4 \theta = 2^{-4}\left[\left(z^4 + z^{-4}\right) - 4\left(z^2 + z^{-2}\right) + 6\right]$$

$$\sin^4 \theta = 2^{-4}\left[2\cos 4\theta - 8\cos 2\theta + 6\right]$$

$$\sin^4 \theta = 2^{-3}\left(\cos 4\theta - 4\cos 2\theta + 3\right)$$

Also when n=6 in equation (6) above;

$$\sin^6 \theta = (2i)^{-6}\left(z - z^{-1}\right)^6$$

$$\sin^6 \theta = -2^{-6}\left(z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}\right)$$

$$\sin^6 \theta = -2^{-6}\left[\left(z^6 + z^{-6}\right) - 6\left(z^4 + z^{-4}\right) + 15\left(z^2 + z^{-2}\right) - 20\right]$$

$$\sin^6 = -2^{-6}\left(2Cos6\theta - 12Cos4\theta + 30Cos2\theta - 20\right)$$

$$\sin^6 = -2^{-5}\left(Cos6\theta - 6Cos4\theta + 15Cos2\theta - 10\right)$$

Also when n=8 in equation (6) above;

$$\sin^8 \theta = (2i)^{-8}\left(z - z^{-1}\right)^8$$

$$\sin^8 = 2^{-8}\left[z^8 - 8z^6 + 28z^4 - 56z^2 + 70 - 56z^{-2} + 28z^{-4} - 8z^{-6} + z^{-8}\right]$$

$$\sin^8 \theta = 2^{-8}\left[\left(z^8 + z^{-8}\right) - 8\left(z^6 + z^{-6}\right) + 28\left(z^4 + z^{-4}\right) - 56\left(z^2 + z^{-2}\right) + 70\right]$$

$$\sin^8 \theta = 2^{-8}\left[2Cos8\theta - 16Cos6\theta + 56Cos4\theta - 112Cos2\theta + 70\right]$$

$$\sin^8 \theta = 2^{-7}\left[Cos8\theta - 8Cos6\theta + 28Cos4\theta - 56Cos2\theta + 35\right]$$

TIJSRAT
ISSN: 2623-7861

Bringing the expanded form of the cosine functions together;

$$\sin^4\theta = 2^{-3}\left(\cos 4\theta - 4\cos 2\theta + 3\right)$$

$$\sin^6 = -2^{-5}\left(Cos6\theta - 6Cos4\theta + 15Cos2\theta - 10\right)$$

$$\sin^8\theta = 2^{-7}\left[Cos8\theta - 8Cos6\theta + 28Cos4\theta - 56Cos2\theta + 35\right]$$

$$\vdots \qquad \vdots \qquad\qquad\qquad \vdots$$

$$\sin^n\theta = 2^{-n}\left({}^nC_{\frac{n}{2}}\right) + 2^{(1-n)}i^n \sum_{k=0}^{\left(\frac{n-2}{2}\right)} (-1)^k\, {}^nC_k Cos(n-2k)\theta$$

Provided that n is even.

Derivation of the Odd order Sine Functions Algorithm

When n=5 in equation (6)

$$\sin^5\theta = 2^{-5}i^4\left(z - z^{-1}\right)^5$$

$$\sin^5\theta = 2^{-5}i^4\left(z^5 - 5z^3 + 10z - 10z^{-1} + 6z^{-3} - z^{-5}\right)$$

$$\sin^5\theta = 2^{-5}\left[\left(z^5 - z^{-5}\right) - 5\left(z^3 - z^{-3}\right) + 10\left(z - z^{-1}\right)\right]$$

$$\sin^5\theta = 2^{-5}\left(2\sin 5\theta - 10\sin 3\theta + 20\sin\theta\right)$$

$$\sin^5\theta = 2^{-4}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right)$$

For n=7

$$\sin^7\theta = 2^{-7}i^6\left(z - z^{-1}\right)^7$$

$$\sin^7\theta = -2^{-7}\left[z^7 - 7z^5 + 21z^3 - 35z + 35z^{-1} - 21z^{-3} + 7z^{-5} - z^{-7}\right]$$

$$\sin^7\theta = -2^{-7}\left[\left(z^7 - z^{-7}\right) - 7\left(z^5 - z^{-5}\right) + 21\left(z^3 - z^{-3}\right) - 35\left(z - z^{-1}\right)\right]$$

$$\sin^7\theta = -2^{-7}\left[2\sin 7\theta - 14\sin 5\theta + 42\sin 3\theta - 70\sin\theta\right]$$

$$\sin^7\theta = -2^{-6}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right]$$

For n = 9

$$\sin^9\theta = 2^{-9} i^{-8}\left(z - z^{-1}\right)^9$$

$$\sin^9 = 2^{-9}\left[z^9 - 9z^7 + 36z^5 - 84z^3 + 126z - 126z^{-1} + 84z^{-3} - 36z^{-5} + 9z^{-7} - z^{-9}\right]$$

$$\sin^9\theta = 2^{-9}\left[\left(z^9 - z^{-9}\right) - 9\left(z^7 - z^{-7}\right) + 36\left(z^5 - z^{-5}\right) - 84\left(z^3 - z^{-3}\right) + 126\left(z - z^{-1}\right)\right]$$

$$\sin^9\theta = 2^{-9}\left[2\sin 9\theta - 18\sin 7\theta + 72\sin 5\theta - 168\sin 3\theta + 252\sin\theta\right]$$

$$\sin^9\theta = 2^{-8}\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]$$

Bringing the series expansions together, it could be observed that;

$$\sin^5\theta = 2^{-4}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

$$\sin^7\theta = -2^{-6}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right]$$

$$\sin^9\theta = 2^{-8}\left[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta\right]$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\sin^n\theta = (2i)^{1-n}\sum_{k=0}^{\left(\frac{n-1}{2}\right)}(-1)^k\,{}^nC_k\,Sin(n-2k)\theta$$ provided n is odd.

Hence, combining both results for odd and even nth power of sine functions in a composite form, we have that;

$$Sin^n\theta = \begin{cases} 2^{-n}\left({}^nC_{\frac{n}{2}}\right) + 2^{(1-n)} i^n \sum_{k=0}^{\left(\frac{n-2}{2}\right)} (-1)^k\ {}^nC_k Cos(n-2k)\theta;\ n = 2,4,6\ldots \\ (2i)^{1-n}\sum_{k=0}^{\left(\frac{n-1}{2}\right)}(-1)^k\ {}^nC_k\ Sin(n-2k)\theta\ ;\ n = 1,3,5\ldots \qquad \forall i = \sqrt{-1}. \end{cases} \tag{9}$$

VAIDATION AND APPLICATION OF THE PROPOSED "ONAHDO" SERIES IN INTEGRAL CALCULUS

APPLICATION OF ODD POWERS SINE FUNCTIONS' ALGORITHM

TIJSRAT
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The above series expansion algorithms proposed by this study was used to integrate the following sine and cosine of higher powers.

Example 1

Evaluate $\int Sin^3\theta d\theta$

Since n=3=odd, we will use the series in equation (9);

Series expansion for $sin^n\theta$when n is odd such that (n=1,3,5,7,9------);

$$sin^n\theta = (2i)^{(1-n)}\sum_{k=0}^{\left(\frac{n-1}{2}\right)}(-1)^k\ {}^nc_k\, sin(n-2k)\theta$$

$$sin^3\theta = (2i)^{-2}\sum_{k=0}^{1}(-1)^k\ {}^3c_k\, sin(3-2k)\theta$$

where i=$\sqrt{-1}$.

$$sin^3\theta = -2^{-2}[sin3\theta - 3sin\theta]$$

The integral of $sin^3\theta\ is$;

$$\int sin^3\theta d\theta = \frac{1}{12}cos3\theta - \frac{3}{4}cos\theta + C$$

Example 2

Evaluate $\int Sin^5\theta d\theta$

Since n=5=odd, we will use the series in equation (9);

$$sin^n\theta = (2i)^{(1-n)}\sum_{k=0}^{\left(\frac{n-1}{2}\right)}(-1)^k\ {}^nc_k\, sin(n-2k)\theta$$

$$sin^5\theta = 2^{-4}[sin5\theta - 5sin3\theta + 10sin\theta]$$

The integral of $sin^5\theta\ is$;

$$\int \sin^5\theta d\theta = -\frac{1}{80}\cos 5\theta + \frac{5}{48}\cos 3\theta - \frac{5}{8}\cos\theta + C$$

Example 3

Evaluate $\int Sin^7\theta d\theta$

Since n=7=odd, we will use the series in equation (9);

TIJSRAT
ISSN: 2623-7861

$$sin^n\theta = (2i)^{(1-n)} \sum_{k=0}^{\left(\frac{n-1}{2}\right)} (-1)^k \, {}^nc_k \, sin(n-2k)\theta$$

where $i=\sqrt{-1}$

$$sin^7\theta = -2^{-6}[sin7\theta - 7sin5\theta + 21sin3\theta - 35sin\theta]$$

The integral of $sin^7\theta$ *is*;

$$\int sin^7\theta d\theta = \frac{1}{448}cos7\theta - \frac{7}{320}cos5\theta + \frac{7}{64}cos3\theta - \frac{35}{64}cos\theta + C$$

Example 4

Evaluate $\int Sin^9\theta d\theta$

Since n=9=odd, we will use the series from equation (9);

$$sin^n\theta = (2i)^{(1-n)} \sum_{k=0}^{\left(\frac{n-1}{2}\right)} (-1)^k \, {}^nc_k \, sin(n-2k)\theta$$

where $i=\sqrt{-1}$

$$sin^9\theta = 2^{-8}[sin9\theta - 9sin7\theta + 36sin5\theta - 84sin3\theta + 126sin\theta]$$

The integral of $sin^9\theta$ *is*;

$$\int sin^9\theta d\theta = -\frac{1}{2304}cos9\theta + \frac{9}{448}cos7\theta - \frac{9}{320}cos5\theta + \frac{7}{64}cos3\theta - \frac{63}{128}cos\theta + C$$

APPLICATION OF EVEN POWERS SINE FUNCTIONS' ALGORITHM

Example 5

Evaluate $\int Sin^2\theta d\theta$

Since n=2=even, we will use the series from equation (9);

$$sin^n\theta = 2^{-n}\left({}^nc_{n/2}\right) + \left(2^{(1-n)}i^n\right) \sum_{k=1}^{\left(\frac{n-2}{2}\right)} (-1)^k \, {}^nc_k \, cos(n-2k)\theta$$

$$sin^2\theta = 2^{-2}(2) + -2^{-1}[cos2\theta]$$

$$sin^2\theta = -2^{-1}[cos2\theta - 1]$$

The integral of $sin^2\theta\ is$;

$$\int sin^2\theta d\theta = \frac{1}{2}\theta - \frac{1}{4}sin2\theta + C$$

Example 6

Evaluate $\int Sin^4\theta d\theta$

Since n=4=even, we will use the series from equation (9);

$$sin^n\theta = 2^{-n}\left({}^nc_{n/2}\right) + \left(2^{(1-n)}i^n\right)\sum_{k=1}^{\left(\frac{n-2}{2}\right)}(-1)^k\,{}^nc_k\,cos(n-2k)\theta$$

$$sin^4\theta = 2^{-3}[cos4\theta - 4cos2\theta + 3]$$

The integral of $sin^4\theta\ is$;

$$\int sin^n\theta d\theta = \frac{1}{32}sin4\theta - \frac{1}{4}sin2\theta + \frac{3}{8}\theta + C$$

Example7

Evaluate $\int Sin^6\theta d\theta$

Since n=6=even, we will use the series;

$$sin^n\theta = 2^{-n}\left({}^nc_{n/2}\right) + \left(2^{(1-n)}i^n\right)\sum_{k=1}^{\left(\frac{n-2}{2}\right)}(-1)^k\,{}^nc_k\,cos(n-2k)\theta$$

$$sin^6\theta = -2^{-5}\,[cos6\theta - 6cos4\theta + 15cos2\theta - 10]$$

The integral of $sin^6\theta\ is$;

$$\int sin^6\theta d\theta = -\frac{1}{192}sin6\theta + \frac{3}{64}cos4\theta - \frac{15}{64}cos2\theta + \frac{5}{16}\theta + C$$

Example 8

Evaluate $\int Sin^8\theta d\theta$

Since n=8=even, we will use the series;

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$$sin^n\theta = 2^{-n}\left({}^nc_{n/2}\right) + \left(2^{(1-n)}i^n\right)\sum_{k=1}^{\left(\frac{n-2}{2}\right)}(-1)^k\,{}^nc_k\,cos(n-2k)\theta$$

$$sin^8\theta = 2^{-7}[cos8\theta - 8cos6\theta + 28cos4\theta - 56cos2\theta + 35]$$

The integral of $sin^8\theta\ is$;

$$\int sin^8\theta d\theta = \frac{1}{1024}sin8\theta - \frac{1}{96}sin6\theta + \frac{7}{128}sin4\theta - \frac{7}{32}sin2\theta + \frac{35}{128}\theta + C$$

Example 9

Evaluate $\int Sin^{10}\theta d\theta$

Since n=10=even, we will use the series;

$$sin^n\theta = 2^{-n}\left({}^nc_{n/2}\right) + \left(2^{(1-n)}i^n\right)\sum_{k=1}^{\left(\frac{n-2}{2}\right)}(-1)^k\,{}^nc_k\,cos(n-2k)\theta$$

$$sin^{10}\theta = -2^{-9}[cos10\theta - 10cos8\theta + 45cos6\theta - 120cos4 + 210cos2\theta - 126]$$

The integral of $sin^8\theta\ is$;

$$\int sin^{10}\theta d\theta = -\frac{1}{5120}sin10\theta + \frac{5}{2048}sin8\theta - \frac{15}{1024}sin6\theta + \frac{15}{256}sin4\theta - \frac{105}{512}sin2\theta + \frac{63}{256} + C$$

Validation of the Series

The series was validated using the Mathematical Induction procedure as used on problems in literature

Theorem 1

The Sinusoidal Onahdo series as represented below, satisfy the Mathematical Induction Axiom;

TIJSRAT
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$$Sin^n\theta = \begin{cases} 2^{-n}\left({}^nC_{\frac{n}{2}}\right) + 2^{(1-n)} i^n \sum_{k=0}^{\left(\frac{n-2}{2}\right)} (-1)^k \ {}^nC_k Cos(n-2k)\theta; \ n = 2,4,6... \\ \\ (2i)^{1-n} \sum_{k=0}^{\left(\frac{n-1}{2}\right)} (-1)^k \ {}^nC_k \ Sin(n-2k)\theta \ ; \ n = 1,3,5... \qquad \forall i = \sqrt{-1}. \end{cases}$$

Proof

Mathematical Induction Proof of the <u>odd power</u> of Sinusoidal Onahdo Series

From the equation below;

$$Sin^n\theta = (2i)^{1-n} \sum_{k=0}^{\left(\frac{n-1}{2}\right)} (-1)^k \ {}^nC_k \ Sin(n-2k)\theta \ ; \ n = 1,3,5... \qquad \forall i = \sqrt{-1}.$$

when n = 1

$$From, \ Sin^n\theta = (2i)^{1-n} \sum_{k=0}^{\left(\frac{n-1}{2}\right)} (-1)^k \ {}^nC_k \ Sin(n-2k)\theta \ ; \ n = 1,3,5... \qquad \forall i = \sqrt{-1}.$$

$$LHS = Sin\theta$$

$$RHS = (2i)^0\left((-1)^0 \ {}^1C_0 Sin(1-2(0)\theta)\right) = 1 \cdot 1 \cdot Sin(1-0)\theta = Sin\theta$$

Thus, since the Left Hand Side (LHS) value is equal to Right Hand Side (RHS) value, then we can remark here that the mathematical Induction assertion holds for the situation when n=1.

when n = K^*

In this case when n = K^*, we assume that the mathematical Induction assertion holds, and hence,

$$\left. Sin^{K^*}\theta = (2i)^{1-K^*} \sum_{k=0}^{\left(\frac{K^*-1}{2}\right)} (-1)^k \ {}^{K^*}C_k \ Sin(K^*-2k)\theta \ ; \ K^* = 1,3,5... \qquad \forall i = \sqrt{-1}. \right\} \qquad (10)$$

when n = $K^* + 1$:

Then, obviously from equation (10) below, we have;

$$Sin^{K^*+1}\theta = (2i)^{1-K^*+1} \sum_{k=0}^{\left(\frac{K^*-1+1}{2}\right)} (-1)^k \; {}^{K^*+1}C_k \; Sin\left(K^* + 1 - 2k\right)\theta \; ; K^* = 1,3,5... \qquad (11)$$

To prove the result in equation (11) we recall from algebra that;

$$Sin^{K^*+1}\theta = Sin^{K^*}\theta \cdot Sin\theta \qquad (12)$$

Hence, putting equation (10) into (12) gives;

$$Sin^{K^*+1}\theta = Sin^{K^*}\theta \cdot Sin\theta = \left((2i)^{1-K^*} \sum_{k=0}^{\left(\frac{K^*-1}{2}\right)} (-1)^k \; {}^{K^*}C_k \; Sin\left(K^* - 2k\right)\theta \right) \cdot Sin\theta$$

Thus, since the Left Hand Side (LHS) value is equal to Right Hand Side (RHS) value, when any value of K^*, then we can remark here that the mathematical Induction assertion holds for the situation when $n = K^* + 1$. $\quad QEQ$

Conflict of Interest

The author hereby declared that there was no conflicting research interest to this study

Conclusion

From all the derivation, analysis and application of the algorithm/series proposed in this study, it has been clearly shown (using all the inference from the theorems of algebra) that the proposed series by this study eases researchers of the stress of using the cumbersome reduction-formula and rigorous substitutions during integration of related family of functions. it was also discovered that the study compares satisfactorily with the existing works in literature. And it has proven to be an easy and alternative means of linearising higher order of Sine functions. Consequently, the results from this study have various use in the field of Engineering and applied Sciences. So also, the algorithm proposed by this study, could be recommended as an alternative approach for solving n^{th} power Sine problems in the world of calculus.

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