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GLOBAL CONVERGENCE OF A HYBRID SMAR-AMRI CONJUGATE GRADIENT METHOD WITH DESCENT PROPERTIES

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ABSTRACT

In this paper, we propose a new conjugate gradient method for solving nonlinear unconstrained optimization. The new method consists a new hybrid form from SMAR and AMRI which search direction has its parameter of Polak and Ribiere (PRP). The proposed method is satisfying the descent condition, sufficient descent condition under exact line search and conjugacy condition. By using several test functions, numerical results show that the proposed method is most efficient compared to some of the existing methods. Most especially the AMRI and SMAR methods. MATLAB was deployed for computation.

KEYWORDS: Unconstrained optimization, Conjugate gradient method, Line search, Descent property, Global convergence.

Introduction

We are interested to consider the unconstrained optimization problem

$$\text{Min}\{ f(x) : x \in R^n \} \quad (1)$$

Where $f : R \to R^n$ is a continuously differentiable function, bounded from below for solving this problem it is well known that there are many methods for solving optimization problems. Where the conjugate gradient CG method is a powerful line search method because of its simplicity and its very low memory requirement. Especially for large scale optimization problems. The following iterative formula is often used by the nonlinear CG method

$$k = 0,1,2,\ldots, \quad (2)$$

Where $d_k \geq 0$ the step length is computed using exact line search by the formula given as

$$f(x_k + \alpha_k d_k) = min f(x_k + \alpha d_k) \quad (3)$$

For (2) where x_k is the current iterative point $\alpha_k > 0$ is a step length and d_k is the search direction designed by

$$d_k = \begin{cases} -g_k, & \text{if} \quad k = 0, \\ -g_k + \beta_k d_{k-1}, & \text{if} \quad k \geq 1, \end{cases} \quad (4)$$

Where $\beta_k \in R$ is a scalar which determines the different conjugate gradient methods and g_k is the gradient of $f(x)$ at the point of x_k The well-known formula for β_k from the computational point of view is the following PRP method.

$$\beta_k^{PRP} = \frac{g_k^T (g_k - g_{k-1})}{\| g_{k-1} \|^2} \qquad (5)$$

where g_{k-1} and g_k are the gradients of $f(x)$ at the point x_{k-1} and x_k respectively, and $\|.\|$ denotes Euclidean norm of vectors. We also denotes $f(x_k)$ by f_k Polak and Ribiere (1969) proved that this method with the line search is globally convergent when the objective function is convex. Also Powell (1984) gave a counter example to show that there exist nonconvex function on which the PRP method does not converge globally even the exact line search is used. He suggested that $\beta\kappa$ should not be less than zero. Considering this suggestion Gilbert and Nocedal(1992) proved that the modified PRP method $\beta_k^* = \max\{0, \beta_k^{PRP}\}$ is globally convergent with the weak-powell(WWP) line search technique and the assumption of sufficient descent condition.

$$g_k^T d_k \leq -C \| g_k \|^2$$

Where C>0 is a constant holds for all k≥ 0

The line search in the Conjugate gtadient algorithms is often based on the standard Wolfe condition.

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \mu \alpha_k g_k^T d_k. \qquad (6)$$

$$g_{k+1}^T \geq \delta g_k^T d_k \qquad (7)$$

Where d_k is a descent direction and $0 < \mu \leq \delta < 1$

Many of the conjugate gradient method are known and an excellent survey of them with special attention on their global convergence is given by Hager and Zhang(2006).

Different conjugate gradient algorithms correspond to difference choice for the scalar parameter β_k. Some of these methods such as Fletcher and Reeves (FR)(1964). Dai and Yuan(DY)(1999) and conjugate Descent(CD)(1987) Proposed by Fletcher and Reeves classified as first category having the common numerator $\|g_k\|2$

$$\beta_k^{FR} = \frac{\| g_k \|^2}{\| g_{k-1} \|^2} \qquad \beta_k^{CD} = -\frac{\| g_k \|^2}{d_{k-1}^T g_{k-1}}$$

$$\beta_k^{DY} = \frac{\| g_k \|^2}{d_{k-1}^T (g_k - g_{k-1})}$$

The above methods have strong convergence properties but they may have modest computational performance. The poor numerical performance of these method is related to the jamming i.e the algorithms can take many short steps without making sufficient progress to the minimum, on the other hand the methods of Polak Ribiere and Polyak(PRP)(1969),Hesteenes and Stiefel(HS)(1952) or Liu and Storey(LS)(1991), These are also classified as second category having the common numerator as follows

$$\beta_k^{HS} = \frac{g_k^T (g_k - g_{k-1})}{d_{k-1}^T (g_k - g_{k-1})} \qquad \beta_k^{PRP} = \frac{g_k^T (g_k - g_{k-1})}{\| g_{k-1} \|^2}$$

$$\beta_k^{LS} = \frac{g_k^T(g_k - g_{k-1})}{-d_{k-1}^T g_{k-1}}$$

The above methods may not generally be convergent but they may have better computational performances .The global convergence properties of the conjugate gradient methods under different line searches have been studied by many researchers, Zoutendijk (1970) proves that FR methods converges globally under exact line search, Powell (1977) however shows that the performance of the FR method is poor by giving a counter example.

AL-Baali(1985), Touati –Ahmed and Storey(1990), Gilbert and Nocedal(1992) have further analyzed the global convergence of algorithms related to the FR method with the Strong Wolfe condition. Also , some modifications of the CG methods includes Andrei(2009), Rivaie et al(2012), Sun and Zhang(2010),Wei Yao and Liu(WYL)(2006), Abashar et al(AMRI)(2014) and Sulaiman et al(SMAR)(2015) Olowo and Aminu(AOS)(2016) as defined as follows

$$\beta_{k+1}^{WYL} = \frac{g_{k+1}^T\left(g_{k+1} - \frac{\|g_{k+1}\|}{\|g_k\|} g_k\right)}{\|g_k\|^2} \quad (6)$$

$$\beta_{k+1}^{AMRI} = \frac{g_{k+1}^T\left(g_{k+1} - \frac{\|g_{k+1}\|}{\|g_k\|} g_k\right)}{\|d_k\|^2} \quad (7)$$

$$\beta_{k+1}^{SMAR} = \frac{g_{k+1}^T\left(g_{k+1} - \frac{\|g_{k+1}\|}{\|g_k\|} d_k\right)}{\|d_k\|^2} \quad (8)$$

$$\beta_k^{AOS} = \frac{|\,|g_k||^2 - \frac{\|g_k\|}{\|g_{k-1}\|}\,|g_k^T g_{k-1}|}{d_{k-1}^T(d_{k-1}^T - g_k)} \quad (9)$$

.

In this section we focus basically on hybrid conjugate gradient methods. These methods are combination of different conjugate gradient algorithms which may be convex or non-convex combination. Their ideas is to use the projection they are mainly proposed in order to avoid the jamming phenomenon , one of the first hybrid conjugate gradient algorithms was introduced by Touati- Ahmed and Storey(1990) , where he combined two CG methods of PRP and FR to produce a new coefficient given as

$$\beta_k^{TS} = \begin{cases} \beta_K^{PRP} = \frac{g_k^T y_k}{\|g_k\|^2}, if\ 0 \le \beta_k^{PRP} \le \beta_k^{FR}, \\ \beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \quad \text{otherwise.} \end{cases} \quad (13)$$

where $y_k = g_k - g_{k-1}$.

The algorithm is globally convergence since the global convergence of the FR and PRP methods have been proved. The algorithm also produces a better performance when compared to the FR, PRP and other CG coefficients.

Another version of the Hybrid method was proposed by Hu and Storey (1991) defined as

$$\beta_k^{Hus} = max\left\{0, min\{\beta_k^{PRP}, \beta_k^{FR}\}\right\} \quad (14)$$

And also produce a better performance and converge globally. Other versions of the CG coefficients have been proposed.

Gilbert and Nocedal (1992) suggested combination between PRP and FR method

$$\beta_k^{GN} = max\left\{-\beta_k^{FR}, min\{\beta_k^{PRP}, \beta_k^{FR}\}\right\}. \quad (15)$$

Dai and Yuan (1998) proposed a family of conjugate methods is given by

$$\beta_k = \frac{\|g_k\|^2}{\lambda\|g_{k-1}\|^2+(1-\lambda)d_{k-1}^T y_{k-1}}, \quad (16)$$

where $\lambda \in [0,1]$ is parameter.

Dai and Yuan (2000b) proposed three parameter family of conjugate gradient methods

$$\beta_k = \frac{\|g_k\|^2-\lambda_k g_k^T g_{k-1}}{\|g_{k-1}\|^2+\mu_k g_k^T d_{k-1}-\omega_k \beta_{k-1} g_{k-1}^T d_{k-2}}, \quad (17)$$

where $\lambda_k \in [0,1]$, $\mu_k \in [0,1]$, $\omega_k \in [0,1-\mu_k]$ are parameters.

Linear combination between LS and CD, Al-Bayati and Al-Kawaz (2012)

$$\beta_k^{LS-CD} = max\left\{0, min\{\beta_k^{LS}, \beta_k^{CD}\}\right\}. \quad (18)$$

Dai and Yuan (2001) combined their algorithm with the Hestense- Stiefel algorithm, suggested two hybrid methods

$$\beta_k^{hDY1} = max\left\{-c\beta_k^{DY}, min\{\beta_k^{HS}, \beta_k^{DY}\}\right\}, \quad (19)$$

$$\beta_k^{hDY2} = max\left\{0 , min\{\beta_k^{HS}, \beta_k^{DY}\}\right\}, \quad (20)$$

where $c = (1-\sigma)/(1+\sigma)$, for standard Wolfe conditions, under the lipschitz continuity of the gradient.

For more reading and recent findings on this method, refer to Dai and Yuan (2001), Andrei (2008a, 2008b), Al-Bayati and Al-Baro (2010), and Lie et al. (2011).

Recently some CG hybrids coefficients were proposed and their global convergence property were established under exact line search and weak wolfe- powell line search. They are defines as follow.

Sulaiman et al (2014) $\beta_k^{PA} = max\left\{0 , min\{\beta_k^{PRP}, \beta_k^{AMRI}\}\right\}$ (21)

Saman Babaie(2012) $\beta_k^{GN} = max\left\{-\beta_k^{FR} , min\{\beta_k^{PRP}, \beta_k^{FR}\}\right\}$ (22)

Li . Xiangrong et al(2011) $\beta_k^{P-W} = max\left\{\{\beta_k^{PRP}, \beta_k^{WYL}\}\right\}$ (23)

Recently, Abubakar et al.(2021; 2022; Diphol;Kaelo;and Tufa;2022 ; Shummin.N(2018) proposed a hybrid three term CG method in which the search direction is generated from the limited memory Broyden-Fletcher-Goldferb-Shanno(LBFGS) Quasi –Newton method. The method satisfies the sufficient descent condition and fulfills the trust region. Also, in Isam; Khalil and Hassan(2022) and Huda and Huda(2022) investigated similar work titled A new conjugate gradient algorithm for unconstrained optimization using difference line search. Both methods satisfies the sufficient descent property and their global convergence property were established

The following are some of this paper's contributions;

(1) Based on the Li, Xiangrong et al and Shummin methods, a new hybrid CG method for solving unconstrained optimization is proposed.

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(2) The search direction of the proposed method satisfies the sufficient descent property with exact line search.
(3) The global convergence of the proposed method is demonstrated using exact line search
(4) Finally, the computational performance of the new method is presented on several standard test problems.

This paper is organized as follow: in section 2, we suggest a new formula for the coefficient β_k and algorithm. In section 3, we show that this new method satisfies the sufficient descent condition and global convergence properties under the exact line search. In section 4, we present the numerical results and we give the conclusion in section 5.

New Hybrid Coefficient

Motivated by the idea of Shummin(2018). and Li. Xiangrong and Zhao(2012) We suggested a hybrid method which is a non-convex function of SMAR and AMRI method defined as

$$\beta_k^{SA} = max\left\{\left\{\beta_k^{AMRI}, \beta_k^{SMAR}\right\}\right\} \quad (24)$$

The following assumptions are often needed to prove the convergence of the nonlinear conjugate gradient methods (see Dai. Y (2000)) Fletcher R (1964), Gilbert J.C (1992) Wei et al (2006), Abashar et al(2014) etc

The Global Convergence

Assumption 1

(i) f is bouned below on the level set R^n and is continous and differentiable in the neighorhood N of the level set $\Omega = \{x \in R^n \mid f(x) \le f(x_0)\}$ at the initial point x_0

(ii) The gradient $g(x)$ is Lipschitz continuous in N, there exists a constant $L > 0$, such that $\|g(x) - g(y)\| \le L\|x - y\|$ for any $x, y \in N.$ (25)

Convergence Analysis

The convergent properties of β_k^{SA} will be studied in this section. We will only show the result of convergence for general CG method. To prove the convergence we assumed that every search direction d_k should satisfy the descent condition.

$$g_k^T d_k < 0 \quad (26)$$

For all k≥0 if there exist a constant C>0 for all k≥0, then the search direction satisfy the Following sufficient descent condition.

$$g_k^T d_k \le -C \| g_k \|^2 \quad (27)$$

The following theorem is very important in establishing sufficient descent condition

Theorem 1

Consider a CG method with the search direction (4) and β_k^{SA} given as (24) then condition (27) hold for all k≥0

Proof

If $k = 0$, then it is clear that $g_0^T d_0 = -C \| g_0 \|^2$. Hence condition (27) holds true. We have to show that for $k \geq 1$, condition (27) will also hold true.

From search direction in equation (4), we have

$$d_{k+1} = -g_{k+1} + \beta_{k+1}^{SA} d_k \quad (28)$$

We multiply both sides of (28) by g_{k+1}, to obtain

$$g_{k+1}^T d_{k+1} = g_{k+1}^T(-g_{k+1} + \beta_k^{SA} d_k) = -||g_{k+1}||^2 + \beta_{k+1}^{SA} g_{k+1}^T d_k \quad (29)$$

For exact line search, $g_{k+1}^T d_k = 0$. Thus, (29) reduces to

$$g_{k+1}^T d_{k+1} = -\| g_{k+1} \|^2 .$$

This implies that d_{k+1} is a sufficient descent direction. Hence, $g_k^T d_k \leq -C \| g_k \|^2$ holds true.■

Numerical Result and Discussion

In this section we report some result of the numerical experiment, it is well known that there exist many new conjugate gradient method which have good properties and good numerical performances, since the given formula is the linear combination of the SMAR and AMRI method We only test Algorithm 1 under exact line search on the test problems used in the related literature. refer to table 4.1. With the given initial points and dimensions. And compare its performance with those of the SMAR method and the AMRI method. We consider $\|g\| < 10^{-6}$ to be stopping criteria For each of the test problems. Four initial point are used. Starting with the point that is near solution and moving to the point that is far away from solution. these four initial points will lead us to test the global convergence and the robustness of our method. All codes are written on MATLAB 7 Subroutine programming. The test results are run on Pentium® dual Core CPU T4300@ 2.10GHZ processor. 4GB for RAM memory and windows 8 professional operating system. The detail result are listed in appendix A and B showing the result based on number of iteration and result based on CPU time respectively.

Table(4.1) List of problems function

Problem	Function
1	Six hump function for two variables
2	Booth function for two variable
3	Trecanni function for two variables
4	Rosenbrock function for two variables
5	Rosenbrock function for four variables
6	Rosenbrock function for ten variables
7	Rosenbrock function for one hundred variables
8	Rosenbrock function for five hundred variables
9	Rosenbrock function for one thousand variables
10	Rosenbrock function for ten thousand variables
11	Extended penalty function for two variables
12	Extended penalty function for four variables
13	Extended penalty function for ten variables
14	Extended penalty function for one hundred variables

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Table 4.2 Comparing different conjugate gradient methods based on CPU Time . When n=2 to 10,000

P	Name	Dimension	SMAR	AMRI	SA
1	Six hump camel	2	0.3993	0.4468	0.4916
2	Booth	2	0.1691	0.1552	0.1528
3	Trecanni	2	0.2769	0.3185	0.2755
4	Rosenbrock	2	0.8896	1.1501	0.8238
5	Rosenbrock	4	0.9125	1.1434	0.8176
6	Rosenbrock	10	1.0534	1.1664	0.8480
7	Rosenbrock	100	1.1609	1.2505	0.9429
8	Rosenbrock	500	1.5825	1.6193	1.2704
9	Rosenbrock	1000	2.2558	2.5576	1.8423
10	Rosenbrock	10000	19.0265	21.4844	16.2621
11	Extended penalty	2	0.6982	0.5173	0.5239
12	Extended penalty	4	0.9422	0.5043	0.5468
13	Extended penalty	10	0.4921	0.4792	0.3428
14	Extended penalty	100	1.1187	0.9587	0.9629
	TOTAL		30.9777	33.7517	26.1034

Table 4.3 Comparing different conjugate gradient methods based on number of iterations . When n=2 to 10,000

P	Name	Dimension	SMAR	AMRI	SA
1	Six hump camel	2	32	36	33
2	Booth	2	12	12	12
3	Trecanni	2	22	25	22
4	Rosenbrock	2	98	128	91
5	Rosenbrock	4	102	128	91
6	Rosenbrock	10	105	128	93
7	Rosenbrock	100	115	125	94
8	Rosenbrock	500	120	123	95
9	Rosenbrock	1000	110	138	97
10	Rosenbrock	10000	108	123	92
11	Extended penalty	2	55	41	41
12	Extended penalty	4	51	30	43
13	Extended penalty	10	39	38	35
14	Extended penalty	100	85	66	76
	TOTAL		1054	1141	913

Table 4.4 Summary of Comparison of different gradient method based on total CPU time and total number of iteration

Method	Total iteration number	Total CPU time
SMAR	1054	30.9777
AMRI	1141	33.7517
SA	913	26.1034

Appendix A and B Show the performance results respectively, these were evaluated using percentage analysis. Clearly, it shows that the new SMAR-AMRI(SA) hybrid method as it was able to solve all the test problems successfully and reach 100%, however, the SMAR method was able to solve about 100% of the test problems, AMRI method was able solve about 100% of the test problems respectively. This shows that our new hybrid method is more effective than the AMRI and SMAR method. Based on CPU time and total number of iterations.

Conclusion.

In this section we present a hybrid conjugate gradient method for solving unconstrained optimization. The global convergence for non-convex function with exact line search is established. The numerical results show that the proposed method (SA) is competitive to the other Methods conjugate gradient method SMAR and AMRI conjugate gradient method used for hybridizing.

For further research, we hope to focus more attention on the stopping criteria. Several of them are known in literature but the task still remains to find out which is more suitable or to formulate a more robust criteria that suite a wide range of problems.

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APPENDIX A
Numerical Result Based on Number of iterations
Using Exact Line Search

NO	FUNCTION	VARIABLES	INITIAL POINT	SA	AMRI	SMAR	
1	SIX HUMP CAMEL	2	(8,8)	8	10	8	
			(-8,-8)	8	10	8	
			(10,10)	8	8	8	
			(-10,-10)	9	8	8	
				33	36	32	
2	BOOTH	2	(10,10)	3	3	3	
			(25,25)	3	3	3	
			(50,50)	3	3	3	
			(100,100)	3	3	3	
				12	12	12	
3	TRECCANI	2	(5,5)	5	5	5	
			(7,7)	5	8	5	
			(50,50)	7	7	7	
			(100,100)	5	5	5	
				22	25	22	
4	ROSENBROCK	2	(13,13)	30	38	22	
			(24,24)	20	20	20	

			(33,33)	19	28	31	
			(35,35)	22	42	25	
				91	128	98	
5	ROSENBROCK	4	(13,13,13,13)	30	38	22	
			(24,24,24,24)	20	20	20	
			(33,33,33,33)	19	28	34	
			(35,35,35,35)	22	42	26	
				91	128	102	
6	ROSENBROCK	10	(13,13,…,13)	31	38	22	
			(24,24,…,24)	20	20	20	
			(33,33,…,33)	20	28	47	
			(35,35,…,35)	22	42	26	
				93	128	105	
7	ROSENBROCK	100	(13,13,…,13)	32	38	22	
			(24,24,…,24)	20	20	20	
			(33,33,…,33)	20	28	47	
			(35,35,…,35)	22	39	26	
				94	125	115	
8	ROSENBROCK	500	(13,13,…,13)	23	39	24	
			(24,24,…,24)	20	20	20	
			(33,33,…,33)	20	25	50	
			(35,35,…,35)	22	39	26	
				95	123	120	
9	ROSENBROCK	1000	(13,13,…,13)	33	38	24	
			(24,24,…,24)	21	20	20	
			(33,33,…,33)	20	44	50	
			(35,35,…,35)	23	36	26	
				97	138	110	
10	ROSENBROCK	10000	(13,13,…,13)	30	36	24	
			(24,24,…,24)	20	20	23	
			(33,33,…,33)	20	34	35	
			(35,35,…,35)	22	33	26	
				92	123	108	
11	EXTENDED PENALTY	2	(100,100)	10	10	16	
			(105,105)	12	12	14	
			(135,135)	9	9	12	
			(200,200)	10	10	13	
				41	41	55	
12	EXTENDED PENALTY	4	(100,100,100,100)	10	10	14	
			(105,105,105,105)	12	9	12	
			(135,135,135,135)	12	12	13	
			(200,200,200,200)	9	9	12	
				43	30	51	
13	EXTENDED PENALTY	10	(100,100,…,100)	6	6	6	
			(105,105,…,105)	10	10	12	
			(135,135,…,135)	10	10	12	
			(200,200,…,200)	9	12	9	
				35	38	39	
14	EXTENDED PENALTY	100	(100,100,…,100)	22	11	12	
			(105,105,…,105)	14	15	35	
			(135,135,…,135)	20	26	25	
			(200,200,…,200)	20	14	13	
				76	66	85	
				913	1141	1054	

APPENDIX B
NUMERICAL RESULT BASED ON CPU TIME
(Using Exact Line Search)

NO	FUNCTION	VARIABLES	INITIAL POINT	SA	AMRI	SMAR
1	SIX HUMP CAMEL	2	(8,8)	0.1924	0.1244	0.1007
			(-8,-8)	0.1001	0.1228	0.0998
			(10,10)	0.0992	0.0996	0.0987
			(-10,-10)	0.0999	0.1000	0.1001
				0.4916	0.4468	0.3993
2	BOOTH	2	(10,10)	0.0372	0.0368	0.0496
			(25,25)	0.0384	0.0378	0.0377
			(50,50)	0.0387	0.0403	0.0411
			(100,100)	0.0385	0.0430	0.0407
				0.1528	0.1552	0.1691
3	TRECCANI	2	(5,5)	0.0627	0.0480	0.0627
			(7,7)	0.0631	0.0367	0.0637
			(50,50)	0.0870	0.0483	0.0876
			(100,100)	0.0627	0.0477	0.0629
				0.2755	0.1807	0.2769
4	ROSENBROCK	2	(13,13)	0.2721	0.3415	0.2021
			(24,24)	0.1808	0.1803	0.1838
			(33,33)	0.1718	0.2532	0.2781
			(35,35)	0.1991	0.3751	0.2256
				0.8238	1.1501	0.8896
5	ROSENBROCK	4	(13,13,13,13)	0.2701	0.3406	0.1986
			(24,24,24,24)	0.1795	0.1803	0.1802
			(33,33,33,33)	0.1701	0.2511	0.3010
			(35,35,35,35)	0.1979	0.3714	0.2327
				0.8176	1.1434	0.9125
6	ROSENBROCK	10	(13,13,…,13)	0.2821	0.3468	0.2013
			(24,24,…,24)	0.1831	0.1834	0.1841
			(33,33,…,33)	0.1825	0.2556	0.4304
			(35,35,…,35)	0.2003	0.3806	0.2376
				0.8480	1.1664	1.0534
7	ROSENBROCK	100	(13,13,…,13)	0.3188	0.3762	0.2224
			(24,24,…,24)	0.2011	0.2042	0.2032
			(33,33,…,33)	0.2038	0.2843	0.4730
			(35,35,…,35)	0.2192	0.3858	0.2623
				0.9429	1.2505	1.1609
8	ROSENBROCK	500	(13,13,…,13)	0.4388	0.5173	0.3199
			(24,24,…,24)	0.2657	0.2633	0.2663
			(33,33,…,33)	0.2628	0.3289	0.6497
			(35,35,…,35)	0.3031	0.5098	0.3466
				1.2704	1.6193	1.5825
9	ROSENBROCK	1000	(13,13,…,13)	0.6286	0.7110	0.4511
			(24,24,…,24)	0.4016	0.3746	0.3810
			(33,33,…,33)	0.3788	0.8134	0.9445
			(35,35,…,35)	0.4333	0.6586	0.4792
				1.8423	2.5576	2.2558
10	ROSENBROCK	10000	(13,13,…,13)	5.2660	6.2961	4.2247
			(24,24,…,24)	3.5527	3.5308	4.1042
			(33,33,…,33)	3.5642	5.9032	6.1034
			(35,35,…,35)	3.8792	5.7543	4.5942
				16.2621	21.4844	19.0265
11	EXTENDED PENALTY	2	(100,100)	0.1306	0.1262	0.2020
			(105,105)	0.1515	0.1511	0.1789
			(135,135)	0.1145	0.1141	0.1528
			(200,200)	0.1273	0.1259	0.1643
				0.5239	0.5173	0.6982
12	EXTENDED PENALTY	4	(100,100,100,100)	0.1286	0.1269	0.4759
			(105,105,105,105)	0.1512	0.1129	0.1519

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				(135,135,135,135)	0.1528	0.1518	0.1625	
				(200,200,200,200)	0.1142	0.1127	0.1519	
					0.5468	0.5043	0.9422	
	13	EXTENDED PENALTY	10	(100,100,…,100)	0.0781	0.0778	0.0780	
				(105,105,…,105)	0.0258	0.1256	0.1497	
				(135,135,…,135)	0.1253	0.1253	0.1504	
				(200,200,…,200)	0.1136	0.1505	0.1140	
					0.3428	0.4792	0.4921	
	14	EXTENDED PENALTY	100	(100,100,…,100)	0.2943	0.1502	0.1630	
				(105,105,…,105)	0.1907	0.2051	0.4438	
				(135,135,…,135)	0.2101	0.4123	0.3376	
				(200,200,…,200)	0.2678	0.1911	0.1741	
					0.9629	0.9587	1.1187	
					26.1034	33.7517	30.9777	