MPRI-JPAS | INTERNATIONAL JOURNAL OF PURE AND APPLIED SCIENCE
VOL. 11 NO.9 JUN-2020 ISSN: 1660-5332

EXISTENCE AND UNIQUENESS OF THE SOLUTION OF NON-LINEAR INTEGRO-DIFFERENTIAL DELAY EQUATION

A. O. NWAOBURU
Department of Mathematics, Rivers State University, Port Harcourt

Abstract

The existence and uniqueness of a delay system of integro-differential equation with subjection to perturbation. This work gave details of existence and uniqueness theorem of a delay system of integro-differential equation, the perturbation condition, and provides periodic solution to such a system which evidently coincide with $\tilde{x}(t) =$ [redacted]

KEYWORDS: *Existence, Solution, Equation, Uniqueness, Differential*

$f(t,\tilde{x}(t))$.

INTRODUCTION

Differential equations to Scientists and Engineers provide dynamics of mathematical models that describe natural phenomena that are abound in their fields. The dynamics of ordinary differential equations are criticized as being naive and unrealistic as they tend to suggest that events in life happen instantaneously. More realistic systems should encompass not only the present but also the past state or states of any given system. Many real life situations present dynamics of delayed actions. Hence, delay differential equation is an equation which expresses the derivatives of the present state x(t) of a physical system in terms of the past states x(t-r), r>0 and other variables. This principle brings about the formulation of functional differential equations which are mostly acceptable today because it appears to permeate various aspects of life and has lately influenced many scholars who work in this area. They are: Jack Hale

INTERNATIONAL JOURNAL – PAS
VOL. 4 NO.1 MAR-2018 ISSN: 1660-5332

[1977], La Salle[1988], Burton [1985], Chukwu [1992], Onwuatu [1993], Driver [1963], etc.

Conditions for the existence and uniqueness of solutions of differential equations are immense importance in the fields of Mathematics, Science and Engineering and do constitute the core contents. Researches on the subject have been intensive and vast literatures abound on the existence and uniqueness of solutions of ordinary differential equations with initial conditions given by:

$$y = f(x,y); \qquad y(x_0) \;\dot{=}\; y_0 \qquad (1)$$

But there are scanty research results on delay systems. However, this paper seeks to investigate the existence and uniqueness of a delayed system subject to perturbation in non-linear integro-differential equations. These systems are applicable in control system feedback, Biological science, disease models and control, etc.

Integro-differential equation and its perturbation

Let E denote the real line. For a real integral n, E^n denotes the space of real n-tuples with the usual Euclidean norm ||.|| and C([a,b], E^n) is the Banach space of continuous functions from [a, b] into E^n with the supremum norm:$||\emptyset|| = \frac{sup}{-1 \le s \le o} /\emptyset(s)/$ (2)

In this research, the state space will be $C([-1,0], E^n)$ with norm defined by $||X|| = max|x(t)|$.

For the function $x: [-h, t] \to E^n$ and t ∈ [O,T], the symbol X_t denotes the function on [-h,o] defined by

$X_t(s) = x(t+s)$; s∈[-h,o],h>o (3)

Consider the system defined by the intergro-differential equation.

$$\dot{x} = A(t)x(t) + B(t)x(t-1) + \int_{-1}^{0} K(t,s)x(t+s)ds \qquad (4)$$

And its perturbation

$$\dot{x} = A(t)x(t) + B(t)x(t-1) + \int_{-1}^{0} K(t,s)x(t+s)ds + f(t,x_t) \qquad (5)$$

The vector function x has its values in E. The matrix function A, B has appropriate dimensions (m×n) and are assured to be continuous on J, J∈E. The matrix function K is also an (n×n) matrix that is continuous on J×[-1,0].

Let h>o.

Let ∅ be an n-vector function which is continuous on [-1,0]. Then, there exist a unique solution of system (4) on J satisfying x(t)≥∅, for t∈[-1,0]. This solution is given as:

$$x(t) = X(t,0)\phi(0) + \int_{-1}^{0}\int_{0}^{w+1} X(t,s)k(s,w-s)dsdw \quad (6)$$

The solution of (5) is given as:

$$x(t) = X(t,0)\phi(0) + \int_{-1}^{0}\int_{0}^{w+1} X(t,s)k(s,w-s)dsdw + \int_{0}^{t} X(t,s)f(t,x_s)ds \quad (7)$$

Where X(t, s) is an (n×n) matrix function satisfying

$$\frac{dX(t,s)}{dt} = A(t)X(t,s) + B(t)X(t-1,s) + \int_{-1}^{0} k(t,w)X(t+w,s)dw \quad (8)$$

For o≤s≤t≤t_1, such that X(t, t) =i, the identity matrix and X(t, s)=o, for t<s.

3. Existence and Uniqueness of Solutions of Delay Equations

State and prove the existence and uniqueness of the solutions for delay differential equations of the form:

$$\dot{x}(t) = F(t,x_t); \quad t > 0 \qquad x_{t_o} = \Psi \quad (9)$$

This is the most general form of systems (4) and (5).

If E^n denotes the n-dimensional Euclidean space.

Let $C = C(-h,o],E^n$ and define

$$C_H = \{\Psi \in C, ||\Psi|| < H\} \quad (10)$$

i.e an open ball in C.

with radius H, x_t is defined by;

$X_t(s) = x(t+s)$ for s∈[-h,o], h>o (11)

Theorem 1: Existence and Uniqueness

Suppose D is an open set in $E\ X\ C$ and $F: D \to E^n$ is continuous and F(t, w) is Lipschizian in $\boldsymbol{\Psi}$ in each compact set of D. If $(t_0,\Psi)\epsilon\ D$, then there exists a unique solution of the delay differential equation (9) with initial values.

U_1 at $t=t_o$ (12)

On [t_o-h,A_o] where A_o is a suitable positive number. Before outlining the proof, take a look at the following enabling tools such as:

(i) Caratheodory conditions

A function $F : D \rightarrow E^n$ is said to satisfy the caratheodory conditions on D, if:

(a) $F(t, \varphi)$ is measurable in t for each fixed φ

(b) $F(t, \varphi)$ is continuous in $\varphi \epsilon B$

(c) For any $(t, \varphi)\epsilon D$, there exist a neighbourhood $N(t, \varphi)$and a Lebesque integrable function M_x∋ such that

$$|F(s, \varphi)| \leq M_x(s), \ (s, \varphi)\epsilon D \tag{13}$$

then, there exists a solution (5) passing through (s, φ).

Suppose D is an open subset of EXB since F is continuous, it satisfies the Caratheodory conditions. Let's now construct the state space; suppose a>o, b>o, are chosen such that the following rectangle is in D, that is:

$$B = \{(t, \varphi), |t - \delta| \leq a, ||\varphi - o|| < by \leq D \tag{14}$$

$$M = \max_{a \leq t \leq b}(a, b), \quad M(t) = \int_{\delta}^{t} m(s)ds < \infty \tag{15}$$

Therefore, let a_1, b_1 be such that $O \leq a \leq a_1, o \leq b \leq b_1, a \leq b_1, t \in [a, b],$ then one can define

$$B_k = \{x \epsilon B(I_\alpha, E^n), \quad x_\delta = \varphi(0)\|x(t) - \varphi_\delta\| \leq k\} \tag{16}$$

Next, show that B_k is a closed, bounded, convex subset of the Banach space $B(I_\infty, E^\wedge)$

Let $x^n \epsilon B_k$ be a subsequence such that $x^n \rightarrow x$ and let $x_\delta^n \Rightarrow \vDash_\delta, x \in B_k.$ since $x \in B_k$

Then, $||x^n(t) - \varphi(o)|| < K \Rightarrow ||x(t) - \varphi(o)|| \leq K_{o,} ||\varphi(o)|| < \infty$

(17)

Also, taking the limit of $||x^n(t) - \varphi(o)|| \leq k$ as $n \rightarrow \infty,$ $x_o = \varphi$ and limit (X_δ^n) = $x_\delta = \varphi$; as $n \rightarrow \infty$

Therefore, $x \epsilon B_K, \ so \ B_k$ is closed.

ii) **Schauder's Fixed Point Theorem**

This is stated as thus:

A continuous map of a compact convex subsets of a Banach space into itself has at least one fixed point. Therefore, the proof of theorem 1 can be outline as thus:

A Existence

Proof:

i) Since D is an open set in E×C and $F: E\times C_n \to E^n$ is continuous from direct integration of system (9); gives

$$X(t) = \varphi(0) + \int_\delta^t F(s, x_s)ds, \quad x_\delta = \varphi \tag{18}$$

ii) $x \epsilon B_k$, Then,

$$\|x(t)\| = \|x(t) - \varphi(o)\| + \|\varphi(o)\| = k + \|\_(z)\| \tag{19}$$

Clearly, B_k is bounded.

iii) Let $y, z \in B_k$ then

$$\|y(t) - \varphi(o)\| \leq k, \|z(t) - \varphi(o)\| \leq k \tag{20}$$

For $\lambda \in [0,1], \|\lambda y + (1-\lambda)z - \varphi(\| = (\lambda y - \varphi_o) + (1-\lambda)z\| \leq \lambda\|y - \varphi_o\| + (1-\lambda)\|Z - \varphi_o\| \leq k, \lambda y + (1-\lambda)z \in B_k$ (21)

Therefore, B_k is convex.

Now, for any $X(t) \in B_k$, lets define the operator $Tx(t)$ by:

$$Tx(t) \cong \varphi(0) + \int_\delta^t F(s, x_s)ds, \quad t \epsilon I_\alpha \tag{22}$$

Thus $Tx_\delta = T\varphi(0)$, from (22)

$$\|Tx_s - \varphi(0)\| = \left\|\int_0^t F(s, x_s)ds\right\|$$

$\leq \int_\delta^t \|F(s, x_s)ds\|ds \leq \left\|\int_\delta^t m(s)ds\right\| \leq |m(t)| \leq k$

(23)

That is, $\|T_{x_\delta}(t) - \varphi(o)\| \leq K \Rightarrow T_x(t) \in B_k$ (24)

Thus, T: $B_k \to B_k$ is a mapping of B_k onto itself as required. T is therefore well defined, it can now be shown that, it is uniformly continuous. Let $X_n \in B_k$, such that $x_n \to x \epsilon B_k$ then due to Caratheodory conditions, $F(t, x_t)$ is continuous in x for each fixed, t. Thus; it gives.

$$F(t, x_t^n) \to F(t, x_t) \text{ as } n \to \infty \text{ for each } t \in I_\infty \tag{25}$$

Since $(t, x_n) \leq m_x(t)$, then Lebesque convergence theorem implies that:

$$\int_\delta^t F(s, x_s^n)ds \to \int_\delta^t F(s, x_s)ds \text{ as } n \to \infty \tag{26}$$

Therefore, T is continuous. Moreover, for any $x \in B_k$, (23) above gives.

$$\|Tx(t) - Tx(c)\| \leq \|-m(c)\|, \text{for all } t, c \epsilon I_a \tag{27}$$

Since m is continuous on I_a being a Lebesque integrable function), it is uniformly continuous. Thus T, is uniformly continuous. Consequently, by Schauder's fixed point theorem there exists a fixed point of T in B_k; that is;

$Tx(t) = x(t); x(t)$ is the fixed point. From (9),

$$Tx(t) = x(t) = \varphi(0) + \int_{\delta}^{t} F(s, x_s)ds, \ \ t \in I_{\infty} \tag{28}$$

Therefore, x(t) is the solution of the integral equation and iposfacto the differential equation: $\dot{x}(t) = F(t, x_t)$

By differentiation of (28), one can observe that x(t) coincides with the condition of (9). This implies that x(t) is the solution of (9) because it satisfies it. Hence, this proves existence.

B Uniqueness

Suppose x(t), y(t) are two solutions of (5) on $[\delta - \alpha, \delta + \alpha]$ $with\ x_{\delta} = \varphi = y_{\delta}$, then

$$x(t) - y(t) = \int_{\delta}^{t} F(s, x_s) - F\big(s, y(s)\big)ds, \ t \geq \delta \tag{29}$$

Let k be the Lipschitz constant of $F(t, \varphi)$ in a compact set containing the trajectories$\{(t, x_t)\}, \{(t, y_t)\}, \ \ t \in I_{\alpha}$. Choose α such that $K_{\alpha} < I_{\alpha}$. Then for t∈I_{α}, gives:

$|x(t) - y(t)| \leq \int_{0}^{t} k|x(s) - y(s)|ds \leq ksup|x(s) - y(s)|$

(30)

$$x(t) = y(t), for\ t \in I_{\alpha}$$

By considering further intervals successively each of length I_{α}, it can similarly be shown that:

$$x(t) = y(t)\ for\ t \in I\alpha$$

$\therefore x(t)$ is unique.

4. Results of Periodic Systems:

Definition 1: Periodic Systems

Consider the general non-linear ordinary differential equation given by:

$$\dot{x}(t) = F\big(t, x(t)\big) \tag{31}$$

With $F : E \times E^n \rightarrow E^n$ continuous

if for al $(t, x) \in E \times E^n\ and\ T > 0$, the relation

INTERNATIONAL JOURNAL – PAS
VOL. 4 NO.1 MAR-2018 ISSN: 1660-5332

$$f(t,x) = f(t+T,x) \tag{32}$$

holds, the system (31) is said to be T- periodic. In general, such a system does not admit a periodic solution unless some conditions are imposed. It is therefore, natural and useful for application to see that such system admits a solution of the form:

$$x(t+T) = x(t) \;\; for\, all\, t \;\in E,\; T > 0 \tag{33}$$

Definition 2: periodic solution

The solution of (23) defined on E, such that

$$x(t+T) = x(t) \tag{33}$$

for all t in E is called T-periodic or T- periodic harmonic solution. Another natural solution is the existence for some integer k>I, of solution X defined on E and such that:

$$x(t+kT) = x(t) \;\; for\, all\, t\, in\, E \tag{34}$$

for some intgers k>I is called sub-harmonic solutions of order K of the T-periodic solution.

Definition 3: consider the linear homogeneous system given by the equation: $\dot{x} = A(t)x,$

$(-\alpha < t < \alpha)$ (35)

Where A(t) is a continuous (n×n) function of it such that A(t+w)=A(t) (36)

For some constants w>o; then (35) is called a periodic system and w is the period of A(t).

The following theorem is a fundamental result from system (35) it conveys that; the fundamental matrix solution of a periodic homogeneous system is also periodic and has an exponential characterization.

Theorem 2: let $\boldsymbol{\Phi} = \boldsymbol{\Phi}$ (t) be a fundamental matrix (35)

Then $\boldsymbol{\Psi}$ = (t) defined by (t) = $\boldsymbol{\Phi}$ (t+w) (—α<t<α) (37)

is also a fundamental matrix for (35). Corresponding to every. Such fundamental matrix, there exist a periodic non-singular n×n matrix, equal to p(t), with period w, and a constant matrix R such that

$$\Phi(t) = p(t)e^{tr} \tag{38}$$

INTERNATIONAL JOURNAL – PAS
VOL. 4 NO.1 MAR-2018 ISSN: 1660-5332

Proof:

Let $\Phi = \Phi$ (t) be a fundamental matrix of (35) and then

$\dot{\Phi}(t) = A(t)\Phi(t) \ (-\alpha < t < \alpha)$ (39)

And so for t =t+w. Also,

$\dot{\Phi}(t+w) = A(t+w)\Phi(t+w) \implies \Phi(t+w) = A(t)\Phi(t+w)$
(40)

(by periodicity of A(t))

$\dot{\Psi}$ (t) =A(t) Ψ(t) (41)

Thus, $\Psi(t) = \Phi$ (t+w) is a solution matrix of (35). It is a fundamental matrix of (34), since the determinant of Ψ at t_o or

det $\Psi(t)|_{t=o}$ = det$(\emptyset(t+w))|_{t=o}$ = det $\emptyset(t) \neq 0$ (42)

Clearly, there exists a non – singular constant n×n matrix such that:

Φ (t+w) = Φ (t)C (43)

Furthermore, there exists a constant, matrix R such that:

$C=e^{wR}$ (44)

Equation (44) has been proved already by coddington and Levinson [chapter 3, page (65-66)]

Combining (43) and (44), yield:

Φ (t+w) = Φ (t)e^{wR} (45)

Now, let p = p(t) be defined by

$p(t) = \Phi(t)e^{-tR}$ (46)

Then, by (44);

p(t+w) = Φ (t+w) $e^{-(t+w)R}$

= Φ (t)$e^{wR}.e^{-(t+w)R}$ = Φ (t)e^{-tR} = Φ (t) (47)

Therefore, p(t) is periodic. Since, (t) and e^{-tR} are non-singular, so also is p(t).

Preposition 1:

If $t_o \in E$ and X is a solution (23) defined on $[t_o, t_o + T]$ and such that;

$x(t_0 + T) = x(t_0)$ (48)

Then, the function x can be extended to E by periodicity and the corresponding extension will be a T- periodic solution of (23).

Proof:

If $t\epsilon\ E, n\epsilon N$ *such that* $t\epsilon[t_0 + (n-1)T, t_0 + nT]$

INTERNATIONAL JOURNAL – PAS
VOL. 4 NO.1 MAR-2018 ISSN: 1660-5332

Hence, t – (n-1) T∈[t_O,t_O + T]

Define; $\tilde{x} = x(t-(n-1)T)$ (49)

With t – (n-1) T∈[t_o,t_o + T]:

Therefore, $\bar{x}$(t) is an extension of X to E by periodicity. Also, the $\bar{x}(t)$ coincides with x(t) on [t_o,t_o+T]. Next, is to show that $\bar{x}$(t) satisfies (23) on E.

$\tilde{x} = \frac{d}{dt}x(t-(x-1)T) = f(t_0-(n-1)T), x(t-(n-1)T)$

(50)

$= f\big(t-(n-1)T, \tilde{x}(t)\big)$ by (41)

$= f\big(t, \tilde{x}(t)\big), by\ (24)\ for\ all\ t\ in\ E$ (51)

Thus, $\bar{x}$(t) satisfies (23) on E. In particular, if one can find a solution of (23) such that

$x(t_0) = x(t_0+T), t_0 = 0 \quad \Longrightarrow x(0) = x(T)$ (52)

Then, the extension by periodicity of (T) solutions furnishes us with T-periodic solution of (23). This suggest that the study of the periodic boundary value problem which will evidently coincide with

$\tilde{x}(t) = f\big(t, \tilde{x}(t)\big)$ (53)

Where, f:[0, T] x $E^n \rightarrow E^n$ is continuous.

Conclusion

Knowing that the solution to an initial value problem is unique is very valuable from both theoretical practical standpoints. If solution were not unique, then we would have to worry about all possible solutions, even when we were doing numerical or qualitative work. Fortunately, the theorems that guarantee uniqueness and existence have been proved in the main result.

References

Burton, T. A. (1985): Stability and Periodic Solutions of Ordinary and Functional Differential Equations, Dover Publications, Mineola, New York, 2005. (Unabridged, slightly corrected version of the original edition published in the series Mathematics in Science and Engineering, Vol. 178, Academic Press, Orlando.)

Chukwu, E. N. (1992): Stability and Time Optimal Control of Hereditary Systems. Academic Press New York.

INTERNATIONAL JOURNAL – PAS
VOL. 4 NO.1 MAR-2018 ISSN: 1660-5332

Driver, R. D. (1963): A Functional Differential System of Neutral Type in a Two Body Problem of Classical Electrodynamics in Nonlinear Differential Equations and Nonlinear Mechanics. Academic Press, New York.

Hale, J. K. (1977): Theory of Functional Differential Equations. Springer Verlag, New York.

La Salle, J. P. (1988): The Stability and Control of Discrete Processes. Applied Mathematical Sciences, 62, Springer Verlag, New York.

Onwuatu, J. (1993): Null Controllability of Nonlinear Infinite Neutral Systems. Kybernetics; Vol. 29, pp. 325-336.