MPRI-JPAS | INTERNATIONAL JOURNAL OF PURE AND APPLIED SCIENCE
VOL. 10 NO.9 DEC-2019 ISSN: 1660-5332

THE SCHWARZ-CHRISTOFFEL TRANSFORMATION AS A CLASSICAL EXAMPLE OF AN ASYMPTOTIC EXPANSION OF CONFORMAL MAPPING.

[1]RAKIYA M.K. ADAMU, [2]ABDULLAHI M. AUWAL, AND [3]ALIYU DANLADI HINA AND
[1]Computer science Department, Federal Polytechnic, Bauchi [2,3]Mathematic and statistics Department, Federal Polytechnic, Bauchi

Abstract

Solutions to the Schwarz-Christoffel integrand equation are conformal maps from the upper half-plane to circular arc polygons, plane regions bounded by straight line segments and arbitrary arcs of circles. The Schwarz-Christoffel transformation provides a basis for computation of conformal maps onto arbitrary domains by approximating the conformal maps by the polygons and solving the associated differential equations numerically, since Schwarz-Christoffel differential equation cannot solved in closed form except in very special cases, when the polygon involved only two or three vertices. The Schwarz-Christoffel transformation allows many physical problems posed on to two dimensional polygonal regions such as heat flow, fluid flow, electrostatics and many more physical problems to be solved numerically. We develop method for numerically integrating this equation. Particular attention is given to the behavior near corner singularities. We also presented [redacted] *and analyzed the used of straightforward integration by parts for asymptotic behavior of Schwarz-Christoffel integrand, the leading behavior of the integrand was analytically obtained and the conditions for existence of the asymptotic expansion of Schwarz-Christoffel integral have been determined.*

KEYWORDS: *Schwarz-christoffel transformation, conformal mapping, asymptotic behavior, polygons, and complex plane.*

Introduction

A conformal mapping is a powerful technique for solving boundary value problems by the use of geometric transformation in the complex plane. Churchill (1990) it is especially useful in finding solutions to Laplace equation in two dimensions.

The term conformal mapping is derived from a geometric property of two dimensional mapping known as conformality; we say that a mapping is conformal if it preserves the angle of intersection between smooth arc. In other words a complex value mapping f is conformal on a domain D in the complex plane if for any z_0 in D and any smooth paths $\alpha(t)$and $\beta\,(t)$ in D that intersect at z_0 with angle of intersection $\varphi,$ the differentiable paths $f(\alpha(t))$and $f\,(\beta(t)),$ which intersect in the image of f at $f\,(z_0)$ will have the same angle of intersection φ. Doughlass (1990).

Although many physical problems are expressed as differential or boundary value surfaces often, these surfaces are or can be approximately by two dimensional polygons. When this occurs, one method of determining accurate solution is by assuming the polygonal domain exists in the complex plane and determining a conformal map, which preserves the structure of Laplace equation that restates the problems in a simple domain. Warner (2008).

Since any solution remains a solution under conformal mapping, a suitable transformation can therefore be used to map a known solution in a simple geometry to geometry of interest. However, Schwarz-christofell transformation is a conformal mapping from the upper half of the complex place to a polygonal domain. It allows many physical problems posed on two dimensions, polygon regions, such as heat flow, fluid flow and electrostatics, to be solved numerically. This type of problem cannot generally be solved in closed form; the Schwarz-christoffel transformation provides an exceptionally accurate method of solution.

The diversity of these methods is in part due to the many connections between conformal mapping and other areas of mathematics. Region with smooth boundaries can be efficiently mapped by many different techniques, but corners often present special difficulties because of singularities in the mapping function. The Schwarz-christoffel transformation and its relatives on the other hand, explicitly take corners into account, Warner (2008). The method of determining the specific transformation is provided by Schwarz-christoffel transformation. Unfortunately the Schwarz-christoffel formula is not easy to evaluate, and required both effective integration analysis, however, the investigation on asymptotic behavior of conformal mapping using Schwarz-christoffel transformation is not a trivial problem and is the subject of this research.

THE SCHWARZ-CHRISTOFFEL TRANSFORMATION

Any conformal map from the upper half plane to a simply connected polygon, with or without corners at infinity, can be expressed in the form;

$$f(z) = A\int_o^z \prod_{j=1}^{n} (\xi - x_j)^{-\theta j/\pi} \, d\xi + B \qquad (1)$$

Known as Schwarz-christoffel formula. In this equation; ξ is an independent complex variable in upper half plane, the θj are the exterior angle of polygon, the x_j are the prevertices of the mapping (given along the real axis), n is the number of vertices of the polygon and A and B are the complex constants that specify the location, size and orientation of the image polygon in the complex plane. The θ_j must satisfy,

$$\sum_{j=1}^{n} \theta j = 2\pi; \qquad (2)$$

This ensures the completeness of the image polygon. Warner (2008). Moreover the Schwarz-christoffel integral is a formula that describes a conformal mapping of the upper half-plane on to the interior of closed "polygon".

We defined a polygon as a connected piece wise smooth path in the plane, which is composed of finite number of straight-line segment, joined in sequence, so that each segment is joined to its predecessor and/or successor in the sequence at a common end points and has no point in common with any other segment. The end point of the segment is called vertices of polygon .If the polygon happens to be simple closed path; we called it a Jordan polygon which means its interiors simply connected. Douglass (1990).

INTERNATIONAL JOURNAL – PAS
VOL. 4 NO.1 MAR-2018 ISSN: 1660-5332

Now consider a polygon in the w-plane having vertices at $w_1, w_2, \dots. w_n$ with corresponding interior angle $\alpha_1, \alpha_2 \dots. \alpha_n$ see diagram below. Let the points $w_1, w_2, \dots. w_n$ map respectively into point $x_1, x_2, \dots. x_n$ on the real axis of the z-plane.

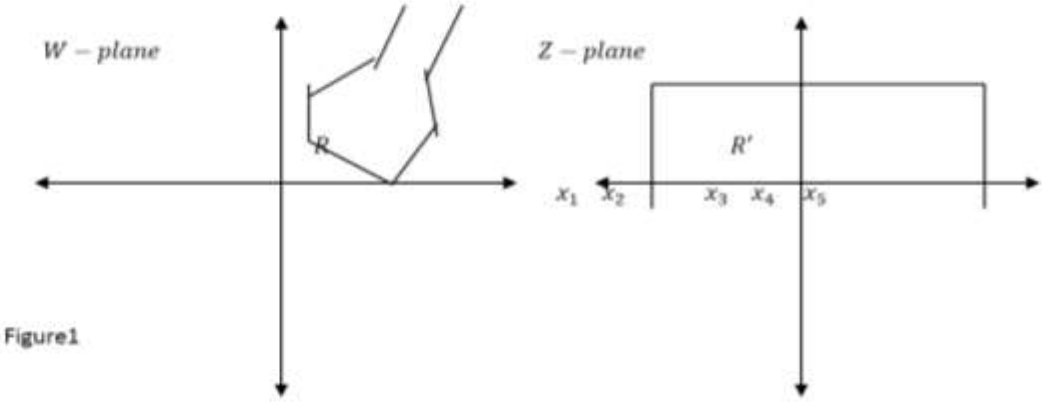

Figure1

There are circumstance in which integration by compound Gauss-Jacobi Quadrature as described in Trefethen (1979) is unsuccessful. This is the case of an integration interval with one endpoint quite near to some prevertex x_j corresponding to a vertex $w_j = \infty$. we cannot evaluate such an integral by considering an interval which begins at x_j, for the integral would then be infinite. The proper approach to this problem is probably the use of integration by parts, which can reduce the singular integrand to one that is not infinite depending on the angle θ_k, one to three applications of integration by pars will be needed to achieve this. We have not implemented this procedure. Trefethen (1979). Nevertheless, integral transform such as the Laplace and Mellin transform, repeated integration by parts will often lead to an asymptotic expansion. Wikipedia the free encyclopedia (2011).

ASYMPTOTIC EXPANSION/ANALYSIS

Asymptotic analysis is that branch of mathematics devoted to the study of the behavior of function at and near given point in their domain of definition. Suppose that $f(z)$ is a function of the complex variable z. suppose further that we wish to study f near the point $z = z_0$ if f is analytic at $z = z_0$ then, the desired behavior can be determined by studying its Tailor series expansion about $z = z_0$ Blesistein and

INTERNATIONAL JOURNAL – PAS
VOL. 4 NO.1 MAR-2018 ISSN: 1660-5332

Handelman (1975). Moreover, it is a remarkable fact that, for large/z/, the behavior of such an $f(z)$ can often be described with enormous accuracy by the partial sums of a different kind of series expansion, called an asymptotic expansion. Carl et al (1966).

THEOREM 1

Suppose that $I(\lambda)$ is defined by $I(\lambda) = \int_a^b h\,(t;\,\lambda)f(t;\lambda)\,dt$ with f independent of λ. suppose further that $f^n(t)$ are continuous for n=0,1,… N+1 while $f^{(N+2)}(t)$ is piecewise continuous in [a,b].finally suppose hat, as $\lambda \to \lambda_o, \left|h^{(-n-1)}(t;\lambda)\right| \leq \alpha_n(t)\emptyset_n(\lambda)$

n= 0,1,……………… N+1 (3)

the functions $\alpha_n(t)$are continuous in [a, b] and function $\emptyset_n(\lambda)$ are elements of an asymptotic sequence then, as $\lambda \to \lambda_o$

$$I(\lambda) \sim \sum_{n=0}^{N} S_n\,(\lambda) \qquad (4)$$

With $S_n(\lambda)$ given by $S_n = (-1)^n\ [f^n\ (b,h)h^{(-n-1)}(b,\lambda) - f^n\ (a,h)h^{(-n-1)}(a,h)]$, represents an asymptotic expansion of $I(\lambda)$ to $N+1$ terms with respect to the auxiliary asymptotic sequence $\{\emptyset_n(\lambda)\}$, simply stated the theorem assert that if f is independent of λ and sufficiently smooth, while the iterated integrals of h are bounded by the terms of an asymptotic sequence, then the integration by parts procedure yield an asymptotic expansion with respect to the sequence.

Proof

In order to establish the theorem we shall show that in $1(\lambda) = \sum_{n=0}^{m} s_n(\lambda) + R_m(\lambda)$, $R_m(\lambda) = 0\left(\emptyset_{m+1}(\lambda)\right) = 0\left(\emptyset_m(\lambda)\right), m = 0,1 \ldots N\ as\ \lambda \to \lambda_o$ for $m \leq N-1$, this is easily accomplished upon integrating by part once more in $R_m\,(\lambda) = (-1)^m \int_\alpha^b f^{m+1}(t;\lambda)h^{(-m-1)}(t;\lambda)dt$ indeed we have

$R_m\,(\lambda) = (-1)^{m+1}\ [f^{m+1}\ (b)h^{(-m-2)}(b,\lambda) - f^{m+1}\ (a)h^{(-m-2)}(a,h)]$
$+ \int_a^b f^{(m+2)}(t;\lambda)$
$h^{(-m-2)}(t;\lambda)dt.$ $m \leq N-1$ (5)

From (3) and (5) we can immediately conclude that;

$R_m\,(\lambda) = 0(\emptyset_{m+1}(\lambda), \quad \lambda \to \lambda_o \quad m \leq N-1)$ (6)

INTERNATIONAL JOURNAL – PAS
VOL. 4 NO.1 MAR-2018 ISSN: 1660-5332

From m = N we start with $R_m(\lambda) = (-1)^{m+1}\int_a^b f^{m+1}(t;\lambda)h^{(-m-1)}(t;\lambda)dt$ and decompose. [a, b] into subintervals throughout each of which $f^{N+2}(t)$ continuous, we then express R_m as a finite sum of integral over these subintervals. Finally, upon integrating by parts once more IN these integrals and upon using (3) we obtain the estimate (6), with m = N this completes the proof. Blesistein and Handelman (1975).

RESULTS

Asymptotic expansion may frequently be obtain by repeated integrations by parts; let us consider the function $f(z)$ defined by integral of equation (1)

(That is, $f(z) = A\int_o^z \prod_{j=1}^n (\xi - x_j)^{-\theta j/\pi}\, d\xi + B$)

One integrations by parts, yield

$$f(z) = A\prod_{j=1}^n [z(z-x_j)^{-\frac{\theta j}{\pi}} + \frac{\theta j}{\pi}\int_o^z \xi(\xi - x_j)^{-\frac{\theta j}{\pi}-1} d\xi + B] \quad (7)$$

Repeating this process N-1 more time yield

$$f(z) = A\prod_{j=1}^n [\sum_{n=0}^{\infty} z^{(n+1)}(z-x_j)^{-\frac{\theta j-n}{\pi}} + \frac{\theta j}{\pi}\int_o^z \xi\left(\xi - x_j)^{-\frac{\theta j}{\pi}-n} d\xi + B\right]$$

(8)

In terms of theorem 1 equation (8) becomes

$$f(z) \sim A\prod_{j=1}^n \left[\sum_{n=0}^{\infty} z^{n+1}(z-x_j)^{-\frac{\theta j-n}{\pi}}\right] + B \quad (9)$$

However, some interesting and informative aspect of the asymptotic expansion of the integral in equation (1) where systematically drive in a fairly wide class.

Furthermore, the method is fundamental to the development of more sophisticated techniques in theorem 1.

Now, we have proposed/developed the asymptotic expansion of the Schwarz –Christoffel

Integral as $z \to \infty$. This is the leading behavior of the integral in equation (1).

And equation (9) motivated the following important questions/notions:

(a) What is the notion when $z \to 0$
(b) What are the notion of arg (z)
(c) What is the notions when $z - x_j = 0$

INTERNATIONAL JOURNAL – PAS
VOL. 4 NO.1 MAR-2018 ISSN: 1660-5332

DISCUSION/ CONCLUSION

In this paper, we have proposed/developed and analyzed the asymptotic behavior of Schwarz-christoffel integral, which describes a conformal mapping of the upper half-plane onto the interior of closed polygon.

Historically speaking, the development of these techniques was motivated by concrete physical problems and in order to decipher the main mathematical and physical features of this asymptotic expansion of the integral it is useful or important to study the notions above, these left for future research. Moreover, it follows that upon use of the straightforward procedure of repeated integrations by parts with respect to theorem 1 can actually yield asymptotic behavior of integral in equation (1). Particularly, if we consider θ_j within a finite interval.

REFERENCES

Ahlfors, L. M., 1979. Complex Analysis .New York: McGraw

Bleistein, N and R, Handelsman., 1997.Asymptotic expansion of integrals. : oxford

Churchill, R. and J, Brown., 1990. Complex variables and application. New York: McGraw-Hill

Christopher, H., 1987. Capillary flows in building elements; sharp front theory and conformal mapping .centre for material science and engineering the University of Edinburgh, the king's buildings, EH93JL: United Kingdom.

Douglass, S.M. and John, D.P.,1990. Complex variable for scientists and engineers. London: collier Macmillan publishers.

Driscoll, T., 1996. A mat lab toolbox for Schwarz-christoffel mapping.ACM Transactions on mathematical software vol.22, : 168-186.

Flory, J., 1986. Conformal mapping –based coordinate generation method for flows in periodic configurations, journal comput.phy62, : 229-265.

Murray, R., 1998. Theory and problems of complex variables with an introduction to conformal mapping and its application SI(metric)edn.

Saff, E. and A, snider.,1990. Fundamentals of complex analysis for mathematics ,science and engineering.

INTERNATIONAL JOURNAL – PAS
VOL. 4 NO.1 MAR-2018 ISSN: 1660-5332

Trefethen, L.; 1979. Numerical computation of the SchwarzChristoffel transformation. Retrieved September 28, 2007, from ftp://reports.stanford.edu/pub/cstr/reports/cs/tr/79/710/CS-TR-79-710.pdf.

Warner, E., 2008.Conformal map using Schwarz-christoffel transformation ;.